A derivation of the number of minima of the Griewank function

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Abstract

The Griewank function is commonly used to test the ability of different solution procedures to find local optima. It is important to know the exact number of minima of the function to support its use as a test function. However, to the best of our knowledge, no attempts have been made to analytically derive the number of minima. Because of the complex nature of the function surface, a numerical method is developed to restrict domain spaces to hyperrectangles satisfying certain conditions. Within these domain spaces, an analytical method to count the number of minima is derived and proposed as a recursive functional form. The numbers of minima for two search spaces are provided as a reference.

Key words: Griewank function, Local minima, Optimization, Multi-modal optimization

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1 Introduction

The Griewank function [1] has been widely used to test the convergence of optimization algorithms [2; 3; 4; 5; 6; 7; 8; 9; 10; 11; 12; 13; 14; 15] because its number of minima grows exponentially as its number of dimensions increases [7; 14]. The function is defined as follows:

$$f_n(\vec{x}) = \frac{1}{4000}\sum_{i=1}^{n} x_i^2 - \prod_{i=1}^{n}\cos\left(\frac{x_i}{\sqrt{i}}\right) + 1$$

within $[-600, 600]^n$ where n is the number of dimensions of the function. The global minimum is located at $\vec{0}$ with a value of 0. The actual number of minima may not be important when global optimization is performed, but it needs to be known to test techniques that search for local optima. Most studies vaguely mention the number of minima of the Griewank function [7; 8; 9], and, to the best of our knowledge, no analytical derivation to determine it has been given in the literature. Knowing the number of minima is critical if the Griewank function serves as the basis for evaluating algorithms designed to find local minima as well as global ones (i.e., multi-modal optimization). In some cases [14], the number of solutions given is inconsistent with analytical results. For example, [14] compared the ability of NichePSO, nbest PSO, lbest PSO, sequential niching, and deterministic crowding based on the number of minima found through numerical searches. However, further work with another algorithm has found a different number of solutions than found by [14]. In order to address this issue and provide a consistent basis for comparing algorithms, this paper analytically derives the number of minima of the Griewank function. We develop an approach in three basic steps. First, we restrict the search space to a hyperrectangle. Second, we show that the hyperrectangle is the maximum possible hyperrectangle of the Griewank function within which local minima on the Griewank function correspond to tangent points on a simpler surface. Third, we develop an analytical approach for counting the number of the tangent points on the simpler surface. This approach yields an accurate count of the number of local minima of the Griewank function within the defined hyperrectangle.

Section 2 elaborates on the characteristics of the function surface and redefines the problem of counting the number of minima to make it analytically tractable. Because of the complex nature of the function surface, the domain space needs to be restricted to hyperrectangles found by the numerical method introduced in Section 2. Although the analytical method to determine the number of minima derived in Section 3 cannot be applied to arbitrary domain spaces, it should be noted that the method does not miss any minima

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within hyperrectangles satisfying certain conditions. As most optimization algorithms are tested within fixed hyperrectangles, it remains practical to use hyperrectangles as domain spaces for testing many optimization algorithms.

2 Redefinition of the problem

The partial derivative of the Griewank function with respect to x_i is

$$\frac{\partial f_n(\vec{x})}{\partial x_i} = \frac{x_i}{2000} + \frac{\sin\left(\frac{x_i}{\sqrt{i}}\right)}{\sqrt{i}} \cdot \prod_{j=1,\, j\neq i}^{n} \cos\left(\frac{x_j}{\sqrt{j}}\right).$$

It is difficult, if not impossible, to analytically solve this non-linear system involving n variables. Global and local minima have to satisfy the following conditions:

$$f'_{n,i}(\vec{x}) = \frac{x_i}{2000} + \frac{\sin\left(\frac{x_i}{\sqrt{i}}\right)}{\sqrt{i}} \cdot \prod_{j=1,\, j\neq i}^{n} \cos\left(\frac{x_j}{\sqrt{j}}\right) = 0 \quad \text{for } i = 1, \cdots, n \tag{1}$$

$$f''_{n,i}(\vec{x}) = \frac{1}{2000} + \frac{1}{i} \cdot \prod_{j=1}^{n} \cos\left(\frac{x_j}{\sqrt{j}}\right) > 0 \quad \text{for } i = 1, \cdots, n \tag{2}$$

where $f'_{n,i}(\vec{x})$ and $f''_{n,i}(\vec{x})$ are the first and second derivatives of $f_n(\vec{x})$, respectively. Note that i is an index for dimensions. Inequality (2) is required to ensure that maxima are not taken into account. By rearranging (2), we obtain $\prod_{j=1}^{n} \cos\left(\frac{x_j}{\sqrt{j}}\right) > -\frac{i}{2000}$. Because the region of non-positive values of $\prod_{j=1}^{n} \cos\left(\frac{x_j}{\sqrt{j}}\right)$ satisfying (1) and (2) (i.e., $f_n(\vec{x}) \geq \frac{1}{4000}\sum_{j=1}^{n} x_j^2 + 1$ at local minima) is outside of the region of its positive values (i.e., $f_n(\vec{x}) < \frac{1}{4000}\sum_{j=1}^{n} x_j^2 + 1$ at local minima), problem domains in this paper are restricted such that

$$\prod_{j=1}^{n} \cos\left(\frac{x_j}{\sqrt{j}}\right) > 0. \tag{3}$$

Since a value of $\frac{i}{2000}$ is small for low dimensions, not much portion of the function space is lost. Eq. (1) can be rewritten as follows:

$$\sin\left(\frac{x_i}{\sqrt{i}}\right) = -\frac{x_i\sqrt{i}}{2000}\left[\prod_{j=1,\, j\neq i}^{n} \cos\left(\frac{x_j}{\sqrt{j}}\right)\right]^{-1} \tag{4}$$

where $\prod_{j=1,\, j\neq i}^{n} \cos\left(\frac{x_j}{\sqrt{j}}\right) \neq 0$ because $\prod_{j=1}^{n} \cos\left(\frac{x_j}{\sqrt{j}}\right) > 0$.

Because $f_n(\vec{x})$ meets the surface $\frac{1}{4000}\sum_{j=1}^{n} x_j^2$ at the global minimum and near local minima, we will find the minima of $f_n(\vec{x})$ by finding the tangent points of $f_n(\vec{x})$ on the simpler surface $\frac{1}{4000}\sum_{j=1}^{n} x_j^2$ and deriving the relationship between these two sets of points. In the following, tangent points refer to the tangent points of the Griewank function on the surface $\frac{1}{4000}\sum_{j=1}^{n} x_j^2$ unless otherwise noted. Since we only want to know the number of minima, their exact coordinates are not of direct interest. In this paper, the number of minima is indirectly derived by counting the number of tangent points associated with them. Because the tangent point associated with the global minimum is the global minimum itself, this method also takes into account the global minimum. Therefore, problem domains have to be carefully defined so that there exists one minimum for each tangent point. As i or x_i increases, $f'_{n,i}(\vec{x})$ also tends to increase along the line $\frac{x_i}{2000}$ and, eventually, no points satisfying (1) are found, which makes global optimization easier [7]. Because there are high correlations between dimensions in high-dimensional problems, it is hard to determine whether or not there are local minima satisfying $0 < \prod_{j=1}^{n} \cos\left(\frac{x_j}{\sqrt{j}}\right) < 1$ by inspecting $f'_{n,i}(\vec{x})$ surfaces separately. It is necessary to know the maximum extent of each x_i beyond which there are no local minima associated with tangent points as shown in Fig. 1. For $n = 1$, it is trivial to check the maximum extent of x_1 because all the points lie on $f'_{1,1}(x_1)$. For $n \geq 2$, a numerical analysis is required to estimate the corners of the hyperrectangle beyond which there exist tangent points not associated with any local minima.

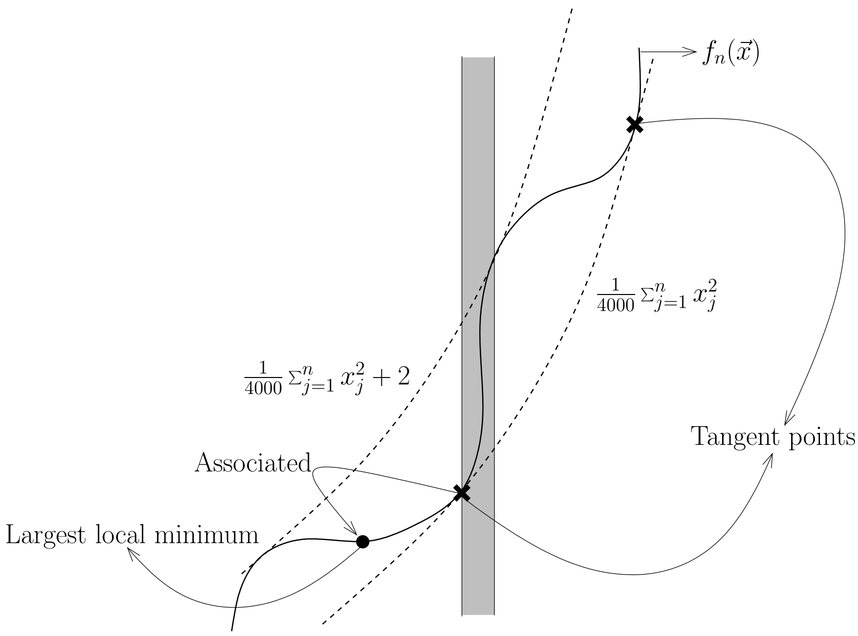

Fig. 1. Out-most region of one dimension of the Griewank function beyond which there exist no more minima. Note that only the tangent point on the left-hand side of the gray region is associated with a local minimum. Problem domains should be smaller than the hyperrectangle defined by the gray region for the method presented in this paper to be valid.

While tangent points are evenly distributed at every $2\pi\sqrt{i}$, local minima are not. If the boundary of a domain space is located between a tangent point and its corresponding local minimum, the number of tangent points is not the same as the number of local minima. For this reason, a problem domain U is defined as $U = (0,\, 2\pi\sqrt{i}k_i)$ where $k_i \in \mathbb{N}$. The maximum value of k_i, $k_{i,\max}$, is defined such that the largest local minimum associated with a tangent point is located in $\left(2\pi\sqrt{i}(k_{i,\max}-1),\, 2\pi\sqrt{i}k_{i,\max}\right)$. Using the periodicity of the sine curve, the ${k_i}^{\text{th}}$ local minimum, $\vec{x}^{k_i} = (x_1^{k_i}, \cdots, x_n^{k_i})$, is obtained by solving the following shifted version of (4):

$$\sin\left(\frac{{x'}_i^{k_i}}{\sqrt{i}}\right) = -\frac{{x'}_i^{k_i}\sqrt{i} + 2\pi i(k_i - 1)}{2000}\left[\prod_{j=1,\, j\neq i}^{n} \cos\left(\frac{x_j^{k_i}}{\sqrt{j}}\right)\right]^{-1} \tag{5}$$

where ${x'}_i^1 = x_i^1$ and ${x'}_i^{k_i} = x_i^{k_i} - 2\pi\sqrt{i}(k_i - 1)$.

For one-dimensional problems, (5) is further simplified by setting $n = 1$ and $\prod_{j=1,\, j\neq i}^{n} \cos\left(\frac{x_j}{\sqrt{j}}\right) = 1$. If the both sides of (5) meet at $x'_i = \frac{3}{2}\pi\sqrt{i}$, there are no local minima at this point because the value of $\prod_{j=1}^{n} \cos\left(\frac{x_j}{\sqrt{j}}\right)$ does not satisfy (3). By solving

$$-\frac{\frac{3}{2}\pi i + 2\pi i(\alpha - 1)}{2000}\left[\prod_{j=1,\, j\neq i}^{n} \cos\left(\frac{x_j}{\sqrt{j}}\right)\right]^{-1} = -1$$

where $\alpha \in \mathbb{R}$, we obtain

$$k_{i,\max} = \lfloor \alpha \rfloor = \left\lfloor \frac{1000}{\pi i} \cdot \prod_{j=1,\, j\neq i}^{n} \cos\left(\frac{x_j}{\sqrt{j}}\right) + \frac{1}{4} \right\rfloor$$

where $\lfloor\cdot\rfloor$ is the maximum integer less than or equal to a given number (i.e., the flooring function). However, since the Griewank function is defined within $[-600, 600]^n$, $2\pi\sqrt{i}k_{i,\max}$ must be less than or equal to $x_{\max} = 600$. Therefore, $k_{i,\max}$ is

$$k_{i,\max} = \min\left(\left\lfloor \frac{x_{\max}}{2\pi\sqrt{i}} \right\rfloor,\, \left\lfloor \frac{1000}{\pi i} \cdot \prod_{j=1,\, j\neq i}^{n} \cos\left(\frac{x_j}{\sqrt{j}}\right) + \frac{1}{4} \right\rfloor\right) \tag{6}$$

and, given a one-dimensional domain space $(0, 2\pi\sqrt{i}k_i)$ where $1 \leq k_i \leq k_{i,\max}$, k_i is the number of local minima.

In problems of more than one dimension, because the position of a local minimum in one axis is highly correlated with those in the other axes, it is not trivial to analytically solve (5) for all dimensions. The values of $\cos\left(\frac{x_i}{\sqrt{i}}\right)$ and $k_{i,\max}$ for $i = 1, \cdots, n$ can be numerically estimated with the pseudo code presented in Fig. 2. The subroutine defined in Fig. 3 is used to solve (5) for each dimension at a time. ${x'}_i^k$ found in this way may not be the correct one because the correlation between dimensions is not taken into account when solving (5). An estimated value of ${x'}_i^k$ is used to evaluate $\prod_{j=1,\, j\neq i}^{n} \cos\left(\frac{x_j}{\sqrt{j}}\right)$, which is iteratively plugged into (5) to estimate the next value of ${x'}_i^k$.

```
Require: n ≥ 1 {Problem dimension}
Require: ε_f {Training threshold for f'_{n,i}(x⃗_{k_{i,max}})}
Require: ε_c {Training threshold for cos(x_i/√i)}
Require: iter_max {Maximum number of iterations for cos(x_i/√i)}
  x_max ⇐ 600 {Initial domain}
  c⃗_out ⇐ 1⃗ {n-tuple output vector}
  repeat
    for i = 1, ··· , n do
      c⃗_tr ⇐ 1⃗ {n-tuple training vector}
      for iter = 1, ··· , iter_max do
        x_i^{k_{i,max}} ⇐ getxi(c⃗_tr, i) in Fig. 3
        c_{tr,i} ⇐ cos( x_i^{k_{i,max}} / √i )
        if iter > 1 and |c_{tr,i} − c_prev| < ε_c then
          break
        c_prev ⇐ c_{tr,i}
        for j = 1, ··· , n, j ≠ i do
          c_{tr,j} ⇐ cos( getxi(c⃗_tr, j) / √j )
      c_{out,i} ⇐ c_{tr,i}
    if |f'_{n,i}(x⃗_{k_{i,max}})| ≤ ε_f for i = 1, ··· , n then
      break
    x_max ⇐ x_max − 2π
  until x_max ≤ 0
  return c⃗_out
```

Fig. 2. Pseudo code to estimate $\cos\left(\frac{x_i}{\sqrt{i}}\right)$ for $i = 1, \cdots, n$.

Once $k_{i,\max}$ is estimated, a problem domain needs to be defined. Define a problem domain by $U = (0,\, x_{i,\max})$, where $0 < x_{i,\max} \leq 2\pi\sqrt{i}k_{i,\max}$, such that $x_{i,\max}$ does not have to be $2\pi\sqrt{i}k_i$ where $1 \leq k_i \leq k_{i,\max}$. When $\prod_{j=1,\, j\neq i}^{n} \cos\left(\frac{x_j}{\sqrt{j}}\right)$ is greater than 0, a local minimum is found in $\left(2\pi\sqrt{i}k_i - \frac{1}{2}\pi\sqrt{i},\, 2\pi\sqrt{i}k_i\right)$ because $\cos\left(\frac{x_i}{\sqrt{i}}\right)$ is greater than 0 satisfying (3), and (4) can hold true only in this range. Likewise, when $\prod_{j=1,\, j\neq i}^{n} \cos\left(\frac{x_j}{\sqrt{j}}\right)$ is less than 0, a local minimum is found in $\left(2\pi\sqrt{i}k_i - \frac{3}{2}\pi\sqrt{i},\, 2\pi\sqrt{i}k_i - \pi\sqrt{i}\right)$. Thus, $x_{i,\max}$ needs to avoid these ranges because, otherwise, it is possible to find local minima not associated with tangent points at $x_i = 2\pi\sqrt{i}k_i \pm \pi\sqrt{i}$, which means that the analytical method introduced in this paper cannot be applied. Therefore, the allowable

Require: $n \geq 1$ {Problem dimension}
Require: $x_{\max} > 0$ {Problem domain}
Require: $\vec{c}_{\text{in}}$ {Input: $\cos\left(\frac{x_i}{\sqrt{i}}\right)$ values}
Require: i {Input: the current training dimension}
Require: All other variables are local ones.
  for $i' = 1, \cdots, n$ **do**
    $\cos\left(\frac{x_{i'}}{\sqrt{i'}}\right) \Leftarrow c_{\text{in},i'}$
  calculate $k_{i,\max}$ according to (6)
  estimate $x'^{k_{i,\max}}_i$ by solving (5)
  $x^{k_{i,\max}}_i \Leftarrow x'^{k_{i,\max}}_i + 2\pi\sqrt{i}(k_{i,\max} - 1)$
  return $x^{k_{i,\max}}_i$

Fig. 3. Pseudo code for the **getxi** subroutine.

range of $x_{i,\max}$ is either

$$\left[2\pi\sqrt{i}k_i - 2\pi\sqrt{i},\, 2\pi\sqrt{i}k_i - \frac{3}{2}\pi\sqrt{i}\right]$$

or

$$\left[2\pi\sqrt{i}k_i - \pi\sqrt{i},\, 2\pi\sqrt{i}k_i - \frac{1}{2}\pi\sqrt{i}\right].$$

The above conditions for $x_{i,\max}$ can be interpreted as

$$0 < x_{i,\max} \leq 2\pi\sqrt{i}k_{i,\max} \tag{7}$$

and

$$x_{i,\max} \in X = \left\{ x_i \;\middle|\; x_i \text{ is a multiple of } \frac{\pi}{2}\sqrt{i} \;\vee\; \left\lfloor \frac{x_i}{\frac{\pi}{2}\sqrt{i}} \right\rfloor \text{ is an even integer} \right\}. \tag{8}$$

In case $x_{i,\max}$ does not satisfy (8) because integer values for $x_{i,\max}$ are preferred, we need to make sure that there are no local minima in

$$\left(\left\lfloor \frac{x_{i,\max}}{\frac{\pi}{2}\sqrt{i}} \right\rfloor \cdot \frac{\pi}{2}\sqrt{i},\, x_{i,\max} \right) \tag{9}$$

where $0 < x_{i,\max} \leq 2\pi\sqrt{i}k_{i,\max}$ and $x_{i,\max} \notin X$. This test can be done indirectly by checking whether or not the distance in the i^{th} axis between $x_{i,\max}$ and the closest tangent point whose coordinate is greater than $x_{i,\max}$ is greater than the possibly largest distance between them. The closest tangent point whose coordinate is greater than x_i is

$$t_i(x_i) = \left\lceil \frac{x_i}{\frac{\pi}{2}\sqrt{i}} \right\rceil \cdot \frac{\pi}{2}\sqrt{i}.$$

Likewise, the largest distance between a local minimum and its corresponding tangent point is obtained by calculating $t_i(x^{k_{i,\max}}_i) - x^{k_{i,\max}}_i$ because $t_i(x^k_i)$ is the tangent point associated with x_i^k, and the distance between them also increases as x_i increases. If $t_i(x_{i,\max}) - x_{i,\max}$ is greater than $t_i(x_i^{k_{i,\max}}) - x_i^{k_{i,\max}}$, there must be one local minimum in $(x_{i,\max}, t_i(x_{i,\max}))$ along the i^{th} axis, which means that there are no local minima in the range defined by (9).

When $x_{i,\max}$ satisfies all the requirements described above, a domain space can be extended to $U = [-x_{i,\max},\, x_{i,\max}]\ \forall i \in \{1, \cdots, n\}$ because the negative domain space $(-x_{i,\max},\, 0)$ is symmetrical to $(0,\, x_{i,\max})$, and the analytical method derived in the following section takes into account both regions implicitly.

3 Derivation of the number of minima

In the previous section, the problem was redefined so that the number of tangent points is the same as the number of minima. The cosine function is defined in $[-1, 1]$ and, thus, the range of the function $\prod_{j=1}^n \cos\left(\frac{x_j}{\sqrt{j}}\right)$ is also restricted to $[-1, 1]$. Consequently, $1 - \prod_{j=1}^n \cos\left(\frac{x_j}{\sqrt{j}}\right)$ has a value in $[0, 2]$ and $f_n(\vec{x})$ in $\left[\frac{1}{4000}\sum_{j=1}^n x_j^2,\ \frac{1}{4000}\sum_{j=1}^n x_j^2 + 2\right]$. Therefore, tangent points of $f_n(\vec{x})$ lie on the surface $\frac{1}{4000}\sum_{j=1}^n x_j^2$ when $\prod_{j=1}^n \cos\left(\frac{x_j}{\sqrt{j}}\right)$ is 1.

The absolute value of $\cos\left(\frac{x_i}{\sqrt{i}}\right)$ is 1 when x_i is a multiple of $\pi\sqrt{i}$. The times $\left|\cos\left(\frac{x_i}{\sqrt{i}}\right)\right|$ equals to 1 depends on the range of x_i or $[x_{i,\min}, x_{i,\max}]$. The number of $\pi\sqrt{i}k$, where $k \in \mathbb{Z}$, within this range is calculated as

$$N_i = \left\lfloor \frac{x_{i,\max}}{\pi\sqrt{i}} \right\rfloor - \left\lceil \frac{x_{i,\min}}{\pi\sqrt{i}} \right\rceil + 1.$$

The number of x_i's satisfying $\cos\left(\frac{x_i}{\sqrt{i}}\right) = 1$ is

$$N_i^+ = \left\lfloor \frac{x_{i,\max}}{2\pi\sqrt{i}} \right\rfloor - \left\lceil \frac{x_{i,\min}}{2\pi\sqrt{i}} \right\rceil + 1$$

and the number of x_i's satisfying $\cos\left(\frac{x_i}{\sqrt{i}}\right) = -1$ is

$$N_i^- = N_i - N_i^+ = \left\lfloor \frac{x_{i,\max}}{2\pi\sqrt{i}} + \frac{1}{2} \right\rfloor - \left\lceil \frac{x_{i,\min}}{2\pi\sqrt{i}} - \frac{1}{2} \right\rceil.$$

Now, the number of maxima and minima can be expressed as $M_n = \prod_{j=1}^n N_j$.

Counting the number of n-tuples in the set

$$A_n = \left\{ \left(\cos\left(\frac{x_1}{\sqrt{1}}\right), \cdots, \cos\left(\frac{x_n}{\sqrt{n}}\right) \right) \in [-1,1]^n \;\middle|\; \prod_{j=1}^{n} \cos\left(\frac{x_j}{\sqrt{j}}\right) = 1 \right\}$$

is a combinatorial problem where combinations take place without repetitions. Any element, $\cos\left(\frac{x_i}{\sqrt{i}}\right)$, of n-tuples belonging to the set A_n must have a value of -1 or 1 because, otherwise, the absolute value of $\prod_{j=1}^{n} \cos\left(\frac{x_j}{\sqrt{j}}\right)$ cannot be 1. Because $\prod_{j=1}^{n} \cos\left(\frac{x_j}{\sqrt{j}}\right)$ should be 1, an even number of elements in an n-tuple have a value of -1, and the other elements have a value of 1. Therefore, the number of n-tuples in the set A_n can be expressed as

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(n-2j)!(2j)!}$$

where $\binom{n}{2j}$ is the binomial coefficient. Encode n-tuples in A_n as $(a_1, a_2, \cdots, a_n)$ where a_i is either 1 or -1. If 1 and -1 are substituted with $+$ and $-$ symbols, respectively, n-tuples in A_n can be represented as $(+, +, \cdots, +)$, $(-, -, +, \cdots, +)$, $(-, +, -, +, \cdots, +)$ (i.e., one n-tuple of $\binom{n}{0}$ and two examples of $\binom{n}{2}$, respectively), and so on. Note that there are an even number of $-$ symbols, and the others are all $+$'s. The numbers of x_i values satisfying $a_i = +$ and $a_i = -$ are N_i^+ and N_i^-, respectively.

Counting all the possible $\vec{x}$ vectors generating n-tuples in the set A_n can be done recursively in terms of n. Let S_n be the number of minima for n-dimensional problems. The simplest form is $S_1 = N_1^+$ for $n = 1$. For $n = 2$, there are S_1 minima when a_2 is fixed to $+$ because S_1 number of x_1's satisfying $\prod_{j=1}^{n-1} a_j = +$ also satisfy $\left(\prod_{j=1}^{n-1} a_j\right) a_n = \prod_{j=1}^{n} a_j = +$. If a_2 is fixed to $-$, $\prod_{j=1}^{n-1} a_j$ must be $-$, and the number of a_1 satisfying this condition is $M_1 - S_1$ (i.e., the number of maxima for $n = 1$). Therefore, $S_2 = S_1 \cdot N_2^+ + (M_1 - S_1) \cdot N_2^-$. Generalizing this recursive form, the following equations are obtained:

$$S_1 = N_1^+ \quad \text{if n=1}, \tag{10}$$
$$S_n = S_{n-1} \cdot N_n^+ + (M_{n-1} - S_{n-1}) \cdot N_n^- \quad \text{if } n > 1 \tag{11}$$

for $[-x_{i,\min}, x_{i,\max}]$ $\forall i \in \{1, \cdots, n\}$. Now, (10) and (11) can be expanded as follows:

$$S_1 = \left\lfloor \frac{x_{1,\max}}{2\pi} \right\rfloor - \left\lceil \frac{x_{1,\min}}{2\pi} \right\rceil + 1 \quad \text{if n=1}, \tag{12}$$

$$\begin{aligned} S_n = S_{n-1} \cdot & \left(\left\lfloor \frac{x_{n,\max}}{2\pi\sqrt{n}} \right\rfloor - \left\lceil \frac{x_{n,\min}}{2\pi\sqrt{n}} \right\rceil + 1 \right) \\ & + \left[\prod_{j=1}^{n-1} \left(\left\lfloor \frac{x_{j,\max}}{\pi\sqrt{j}} \right\rfloor - \left\lceil \frac{x_{j,\min}}{\pi\sqrt{j}} \right\rceil + 1 \right) - S_{n-1} \right] \\ & \times \left(\left\lfloor \frac{x_{n,\max}}{2\pi\sqrt{n}} + \frac{1}{2} \right\rfloor - \left\lceil \frac{x_{n,\min}}{2\pi\sqrt{n}} - \frac{1}{2} \right\rceil \right) \quad \text{if } n > 1 \end{aligned} \tag{13}$$

for $[-x_{i,\min},\, x_{i,\max}]\ \forall i \in \{1, \cdots, n\}$.

4 Results and discussion

Fig. 4 and Table 1 present the maximum estimated number of local minima, $k_{i,\max}$, and the largest local minimum, $x_i^{k_{i,\max}}$, on the i^{th} axis. They define hyperrectangles within which (12) and (13) can be applied. Outside these regions, the analytical method presented in this paper cannot be used to count the number of minima. Fig. 4 shows $k_{i,\max}$ for different dimensions. For $n \geq 43$, the numerical algorithm in Fig. 2 experienced difficulties in finding $k_{i,\max}$, and no plots were drawn. This result might be caused by reducing the search space by 2π in all directions. However, because the number of minima within only a small fraction of hyperrectangles defined by $x_i^{k_{i,\max}}$ is so high even for $n = 3$ (e.g., 1,215 minima in $[-28, 28]^3$, a subspace of $[-x_i^{k_{i,\max}},\, x_i^{k_{i,\max}}]\ \forall i \in \{1, 2, 3\}$), it would be practically enough to define domain spaces for up to $n = 40$. Table 1 shows $k_{i,\max}$ and $x_i^{k_{i,\max}}$ estimated for up to three-dimensional problems. Note that $k_{i,\max}$ for the same i varies with n because of the correlation between dimensions. When defining a domain space by $U = [-x_{i,\max},\, x_{i,\max}]\ \forall i \in \{1, \cdots, n\}$, we need to make sure $0 < x_{i,\max} \leq t_i(x_i^{k_{i,\max}})$. This condition satisfies (7) because $t_i(x_i^{k_{i,\max}}) = 2\pi\sqrt{i}k_{i,\max}$ for all the cases in Table 1. Also, $x_{i,\max}$ has to satisfy (8) or (9).

As a set of examples, domain spaces $U = [-x_{\max},\, x_{\max}]^n$ were evaluated for $1 \leq n \leq 3$ where $x_{\max} \in \{14, 28\}$. Note that, for the sake of simplicity, domain spaces were chosen such that all $x_{i,\max} = x_{\max}$. For $x_{\max} = 14$, (8) holds true when $i = 1$ or 2. The closest tangent point whose coordinate is greater than $x_{3,\max} = 14$ is $t_3(14)$, and the distance in the 3^{rd} axis between $x_{3,\max}$ and $t_3(14)$ is $t_3(14) - 14 = 2.32$. This distance is greater than $t_3(x_3^{k_{3,\max}}) - x_3^{k_{3,\max}} = 1.24$ for $n = 3$ as shown in Table 1. This means that the local minimum associated with $t_3(x_{3,\max})$ exists in $(x_{3,\max}, t_3(x_{3,\max}))$, not in the range defined by (9) for $x_{3,\max} = 14$. For $x_{\max} = 28$, (8) holds true when $i = 2$ or 3. A visual inspection

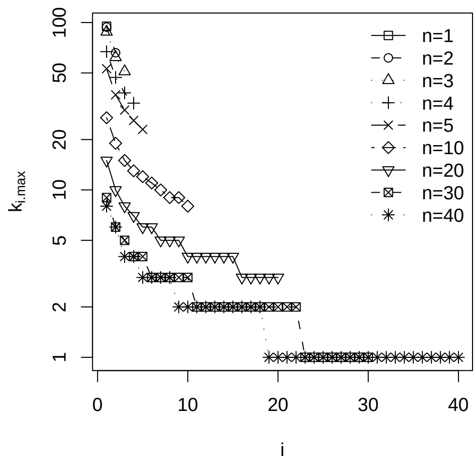

Fig. 4. $k_{i,\max}$ versus i for different problem dimensions.

Table 1
Maximum estimated number of local minima and the largest local minimum in each dimension. $k_{i,\max}$ is the maximum estimated number of local minima on the i^{th} axis within $x_i \in (0, 600)$; $x_i^{k_{i,\max}}$ is the largest local minimum on the i^{th} axis; $t_i(x_i^{k_{i,\max}})$ is its corresponding tangent point; and $t_i(x_i^{k_{i,\max}}) - x_i^{k_{i,\max}}$ is the largest distance in the i^{th} axis between them.

n	i	$k_{i,\max}$	$x_i^{k_{i,\max}}$	$t_i(x_i^{k_{i,\max}})$	$t_i(x_i^{k_{i,\max}}) - x_i^{k_{i,\max}}$
1	1	95	596.60	596.90	0.30
2	1	94	590.28	590.62	0.34
	2	66	585.82	586.46	0.64
3	1	88	552.45	552.92	0.47
	2	62	550.04	550.92	0.88
	3	51	553.78	555.02	1.24

of the x_1 axis and a numerical analysis show that there are no local minima in the range defined by (9) for $x_{1,\max} = 28$. Because $x_{\max} \in \{14, 28\}$ satisfies the boundary conditions specified by (8) and (9), we can safely use (12) and (13) to calculate the number of minima of the Griewank function. Table 2 shows the numbers of minima for the two search spaces for up to three dimensions.

Table 2
Numbers of minima for $[-14, 14]^n$ and $[-28, 28]^n$.

n	$[-14, 14]^n$	$[-28, 28]^n$
1	5	9
2	31	111
3	157	1,215

5 Conclusions

It is difficult to analytically solve the derivative of the Griewank function and directly count the number of minima because of the complex nature of the function surface. The problem of counting the number of minima was redefined as counting the number of tangent points lying on a parabolic surface. A numerical method was developed to find hyperrectangles within which this approach can be applied, and the number of minima of the function was analytically derived within these domain spaces based on a recursive functional form. The maximum extents of hyperrectangles for up to three dimensions were estimated, and the numbers of minima for two search spaces were provided as a reference.

The numerical and analytical methods introduced in this paper can be used to determine the exact number of minima within the domain space defined by a hyperrectangle satisfying certain conditions. The number of minima derived in this paper can serve as a sound basis for evaluating multi-modal optimization algorithms.

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References

[1] A. O. Griewank, Generalized descent for global optimization, Journal of Optimization Theory and Applications 34 (1) (1981) 11–39.

[2] J. Kennedy, Stereotyping: Improving particle swarm performance with cluster analysis, in: Proceedings of the Congress on Evolutionary Com-

putation, IEEE Service Center, Piscataway, New Jersey, 2000, pp. 1507–1512.

[3] T. Krink, J. S. Vesterstrøm, J. Riget, Particle swarm optimisation with spatial particle extension, in: Proceedings of the Congress on Evolutionary Computation, Vol. 2, Honolulu, Hawaii, 2002, pp. 1474–1479.

[4] J. Riget, J. S. Vesterstrøm, A diversity-guided particle swarm optimizer–the ARPSO, Tech. Rep. 2002-02, Department of Computer Science, Aarhus Universitet, Bgn. 540, Ny Munkegade DK-8000 Aarhus C, Denmark (2002).

[5] X.-F. Xie, W.-J. Zhang, Z.-L. Yang, A dissipative particle swarm optimization, in: Proceedings of the Congress on Evolutionary Computation, Honolulu, Hawaii, 2002, pp. 1456–1461.

[6] R. Brits, A. P. Engelbrecht, F. van den Bergh, Scalability of Niche PSO, in: Proceedings of the 2003 IEEE Swarm Intelligence Symposium, 2003, pp. 228–234.

[7] M. Locatelli, A note on the Griewank test function, Journal of Global Optimization 25 (2003) 169–174.

[8] N. Khemka, C. Jacob, Exploratory toolkit for evolutionary and swarm-based optimization, in: Proceedings of the 6th International Mathematica Symposium, Banff, Alberta, Canada, 2004.

[9] S. He, Q. H. Wu, J. Y. Wen, J. R. Saunders, R. C. Paton, A particle swarm optimizer with passive congregation, BioSystems 78 (2004) 135–147.

[10] A. Acan, A. Gunay, Enhanced particle swarm optimization through external memory support, in: Proceedings of the Congress on Evolutionary Computation, Vol. 2, 2005, pp. 1875–1882.

[11] C. K. Monson, K. D. Seppi, Bayesian optimization models for particle swarms, in: Proceedings of the Genetic and Evolutionary Computation Conference, ACM Press, New York, New York, 2005, pp. 193–200.

[12] M. Meissner, M. Schmuker, G. Schneider, Optimized particle swarm optimization (OPSO) and its application to artificial neural network training, BMC Bioinformatics 7 (125).

[13] C. K. Monson, K. D. Seppi, Adaptive diversity in PSO, in: Proceedings of the Genetic and Evolutionary Computation Conference, ACM Press, New York, New York, 2006, pp. 59–66.

[14] R. Brits, A. P. Engelbrecht, F. van den Bergh, Locating multiple optima using particle swarm optimization, Applied Mathematics and Computation 189 (2007) 1859–1883.

[15] X. Wang, X. Z. Gao, S. J. Ovaska, A hybrid optimization algorithm based on ant colony and immune principles, International Journal of Computer Science & Applications 4 (3) (2007) 30–44.